Detailed description of the closing condition and the process for establishing the need for price increases

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1 The closing condition

The third stage of the auction closes when there is at least one Potentially Winning Combination of Bids that includes a bid from every bidder.

Let B denoted the number of B lots available in the third auction stage. The number of lots in each exemption lot category included in the third stage of the auction is equal to the number of winners of B lots minus one, or zero if there are no winners of B lots.

Let $\beta_j^k = (P_j^k, b_j^k, c1_j^k, c2_j^k, c3_j^k)$ be bid k from bidder $j = 1, ...N$ where N is the number of bidders taking part in the third auction stage, P_j^k is the amount of the bid, b_j^k is the number of B lots included in the package of the bid, and c_j^k for $c = c1, c2, c3$ indicates whether the package includes an exemption for the respective Coverage Area Group (depending on the number of Coverage Area Groups that are included in the third auction auction stage). K_j denotes the number of bids submitted by bidder j.

A combination of bids is defined by $x = (x_1^1, x_1^2, ...x_1^{K_1}, x_2^1, ..., x_N^{K_N})$, where x_j^k is a binary variable that is equal to 1 if bidder j's k-th bid is included in the combination and zero otherwise.

In order to check whether the closing condition is met, we define the following:

A **Supply Scenario** S_i with $i= 0, ..., N-1$ is a scenario in which the supply of exemptions for each Coverage Area Group that is included in the third auction stage is fixed at i. With N bidders taking part in the third auction stage, $N-1$ is the maximum number of exemptions that might be available.

A **Feasible Combination of Bids** in a Supply Scenario S_i is a combination of bids that includes at most one bid from each bidder and can be accommodated with the given supply in that scenario. Thus, a Feasible Combination of Bids in Supply Scenario S_i is defined as a combination of bids that satisfies the following conditions:

$$\sum_{j=1}^{N} \sum_{k=1}^{K_j} x_j^k b_j^k \leq B \tag{1.1}$$

$$\sum_{j=1}^{N}\sum_{k=1}^{K_j} x_j^k c_j^k \leq i \; \forall \, c = c1, c2, c3 \tag{1.2}$$

$$\sum_{k=1}^{K_j} x_j^k \leq 1 \; \forall \, j = 1, ..., N \tag{1.3}$$

The **Value** of a Feasible Combination of Bids is denoted as

$$V(x) = \sum_{j=1}^{N}\sum_{k=1}^{K_j} x_j^k P_j^k + \left(B - \sum_{j=1}^{N}\sum_{k=1}^{K_j} x_j^k b_j^k \right) r$$

where r is the reserve price of B lots.

The set X_i^* of **Highest-Value Combinations of Bids** x_i^* in a Supply Scenario S_i is the set of Feasible Combinations of Bids that achieve the highest value, i.e.

$$X_i^* = \left\{ x_i^* : \arg\max_x V(x) \mid S_i \right\}$$

subject to conditions 1.1 to 1.3.

A Feasible Combination of Bids x_i in a Supply Scenario S_i is **Exemption-Compatible** if:

$$\max \left[\left(\sum_{j=1}^{N} z_{ij}^k x_{ij}^k \right) - 1, 0 \right] \geq i$$

where $z_{ij}^k = 1$ if the bid from bidder j includes one or more B lots, and zero otherwise.

Let $\hat{X}_i$ denote the set of all Combinations of Bids in Supply Scenario S_i that are Exemption-Compatible.

Let $\hat{X}_i{}^*$ denote the set of the Highest-Value Combination of Bids in Supply Scenario S_i that are Exemption-Compatible, so $\hat{X}_i^* = X_i^* \cap \hat{X}_i \; \forall i = 0, ..., N-1$.

Let $\hat{X}^*$ denote the set of all the Exemption-Compatible, Highest-Value Combination of Bids across all scenarios, so $\hat{X}^* = \bigcup_{i=0}^{N-1} \hat{X}_i^*$.

The **Potentially Winning Combinations of Bids** X^* is the set of Exemption-Compatible Highest-Value Combinations of Bids across all Supply Scenarios with the highest value, i.e.

$$X^* = \left\{ \hat{x}_i^* : \arg\max_x V(\hat{x}_i^*) \mid x \in \hat{X}^* \right\}$$

Using these definitions, the third stage of the auction closes when the following conditions holds:

$$\exists x \in X^* : \sum_{j=1}^{N}\sum_{k=1}^{K_j} x_j^k = N$$

2 Identifying lot categories for which prices need to increase

If the third auction stage does not close, each of the Potential Winning Combinations of Bids omits all bids from at least one bidder. We call such a bidder an **Omitted Bidder**.

In order to identify Omitted Bidders, we divide the bidders into two distinct sets:

- The first set contains all bidders who are included in each and every Potentially Winning Combination.
- All other bidders fall into the second set (the set of Omitted Bidders).

For notational convenience, let us re-label bidders so that all bidders $j = 1, .., J-1$ are in the first set, and all bidders $j = J, ..., N$ are in the second set, i.e. are Omitted Bidders. The order of bidders within each set is random.

Let β_j^h denote the headline bid submitted by bidder j in the most recent round with lot prices being p_b and p_c with $c = c1, c2, c3$.

In order to identify the lot categories for which prices need to increase, we examine for each omitted bidder $n = J, ..., N$ the potential reason why that bidder's headline bid could not be accommodated. Specifically, we run through the following steps (terminating the process early if we arrive at a point at which we have identified all lot categories as needing a price increase):

1. Set $n = J$.
2. Identify the Potentially Winning Combinations of Bids having replaced $\beta_n^h = (P_n^h, b_{n,}^h, c1_n^h, c2_n^h, c3_n^h)$ with a hypothetical bid $\beta' = (P_n^h - \sum c_n^h p_c, b_n^h, 0, 0, 0)$. If bidder n would still be omitted, the price of B lots needs to increase.
3. Considering each of the available exemption lot categories $c1, c2, c3$ for which $c_n^h = 1$, in turn:
   (a) identify the Potentially Winning Combinations of Bids having replaced β_n^h with a hypothetical bid β' in which the respective c_n^h is retained, the demand for all other lot categories is set to zero and the bid amount is reduced accordingly;
   (b) if bidder n would still be omitted, the price of exemption lots in category c needs to increase.
4. If β_n^h includes bids for more than one exemption lot category and neither the first nor the second step have identified the need for a price increase, identify the Potentially Winning Combinations of Bids having replaced β_n^h with a hypothetical bid $\beta' = (P_n^h - p_b, 0, c1_n^h, c2_n^h, c3_n^h)$. If bidder n would still be omitted, the price of exemption lots in all categories included in the bidder's headline bid needs to increase.
5. If after steps 2 to 4, none of the lot categories have been identified as requiring a price increase, the prices of B lots and exemptions in all Coverage Area Groups for which $c_n^h = 1$ need to increase.
6. If $n < N$, set $n = n+1$ and repeat steps 2 to 5 and perform the tests if the corresponding lot category has not already been identified as requiring a price increase. Otherwise, terminate.